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Abstract. We introduce the notion of a bisector field, which is a maximal collection of pairs of lines such that for each line in each pair, the midpoint of the points where the line crosses every pair is the same, regardless of choice of pair. We use this to study asymptotic properties of pencils of affine conics over fields and show that pairs of lines in the plane that occur as the asymptotes of hyperbolas from a pencil of affine conics belong to a bisector field. By including also pairs of parallel lines arising from degenerate parabolas in the pencil, we obtain a full characterization: Every bisector field arises from a pencil of affine conics, and vice versa, every nontrivial pencil of affine conics is asymptotically a bisector field. Our main results are valid over any field of characteristic other than 2 and hence hold in the classical Euclidean setting as well as in Galois geometries.

Mathematics Subject Classification. Primary 52C30, 51N10.

Keywords. Pencil of conics, Affine conic, Bisector field, Quadrilateral.

1. Motivation

This article focuses on the question of which pairs of lines occur as the asymptotes of hyperbolas in a pencil of affine conics. We were led to this question from several directions because the solution we give to it, the bisector fields of the title, appeared for us as a solution to other seemingly unrelated questions, and the connection between this idea and asymptotes of hyperbolas in pencils seems not to appear in the literature. In this paper, we introduce bisector fields and establish their connection to pencils of conics. In sequels to this article we consider other applications of bisector fields, some of which we describe later in this section in order to help motivate the concept.

Beltrami [1, Section 2] (but see also [7]) dealt with what can be considered a projective version of our asymptotes question. We mention it here by way of contrast with our results. Given a pencil of projective conics and a line ℓ in the projective plane, Beltrami shows the envelope of the pairs of lines that are tangent to a conic in the pencil at the points where ℓ crosses the conic is a rational quartic curve with three cusps, possibly two of which are imaginary. Thus, with ℓ designated the line at infinity and everything else restricted to the resulting affine chart, the asymptotes of the hyperbolas in the pencil are the lines tangent to a rational quartic curve with three cusps, at least one of which is real. The cusps of this envelope are visible in Fig. 1a, b.

Our approach ultimately does give a different proof of Beltrami's theorem, but we postpone the connection with his tricuspidal quartic curve to the article [6], where, using the results of the present paper and projective duality, we arrive at his theorem from a rather different and more general starting place involving planar arrangements of lines. The aim in the present article is to give a very different type of criterion for when pairs of lines occur as asymptotes of hyperbolas in a pencil of affine conics, a criterion that involves only how the lines cross each other. In our approach, the underlying field $\Bbbk$ seldom matters, other than that it has characteristic not equal to 2. (We avoid characteristic 2 because we work with midpoints.)

Our main theorem is that whether a set $\mathbb{A}$ of pairs of intersecting lines in the affine plane over $\Bbbk$ is the set of asymptotes for the hyperbolas in a pencil is detectable from an unusual feature of the geometry of the pairs, namely that each line in each pair simultaneously bisects all other pairs in the set $\mathbb{A}$. What bisects means is that the midpoint of the points where the line crosses a pair of lines is the same regardless of the pair that is chosen from $\mathbb{A}$.

To state the main theorem more precisely, we need to consider not only asymptotes of hyperbolas from a pencil but pairs of parallel lines that share a midline with a pair of parallel lines in the pencil. Taken together, the asymptotes and the pairs of parallel lines are the degenerate conics in an affine net generated by the pencil and the line at infinity. We call such a collection of pairs of lines an *asymptotic pencil* since it represents the asymptotic behavior of a pencil of affine conics. The main theorem of the article is then that a set of pairs of lines is a nontrivial asymptotic pencil if and only if it is a *bisector field*, a collection $\mathbb{B}$ of pairs of lines such that each line in each pair simultaneously bisects all other pairs in the set $\mathbb{B}$ and such that $\mathbb{B}$ is maximal with respect to this bisection property—maximal in the sense that $\mathbb{B}$ cannot be enlarged without destroying the bisection property. Figures 1 and 2 give examples of pencils and their resulting bisector fields.

By way of further motivation we mention several other applications of these ideas that are worked out in sequels to this paper.

(1) Continuing to work over a field of characteristic $\neq 2$, in [5] we study the bisectors of a quadrilateral, where a line is said to bisect a quadrilateral if it bisects both pairs of opposite sides. The collection of all bisectors of a quadrilateral admits a pairing given by an orthogonality relation from an inner product induced by the quadrilateral itself. With this pairing, rather surprisingly, the bisectors form a bisector field—surprising because of the non-obvious symmetry here: although the lines were chosen to bisect the quadrilateral, every pair

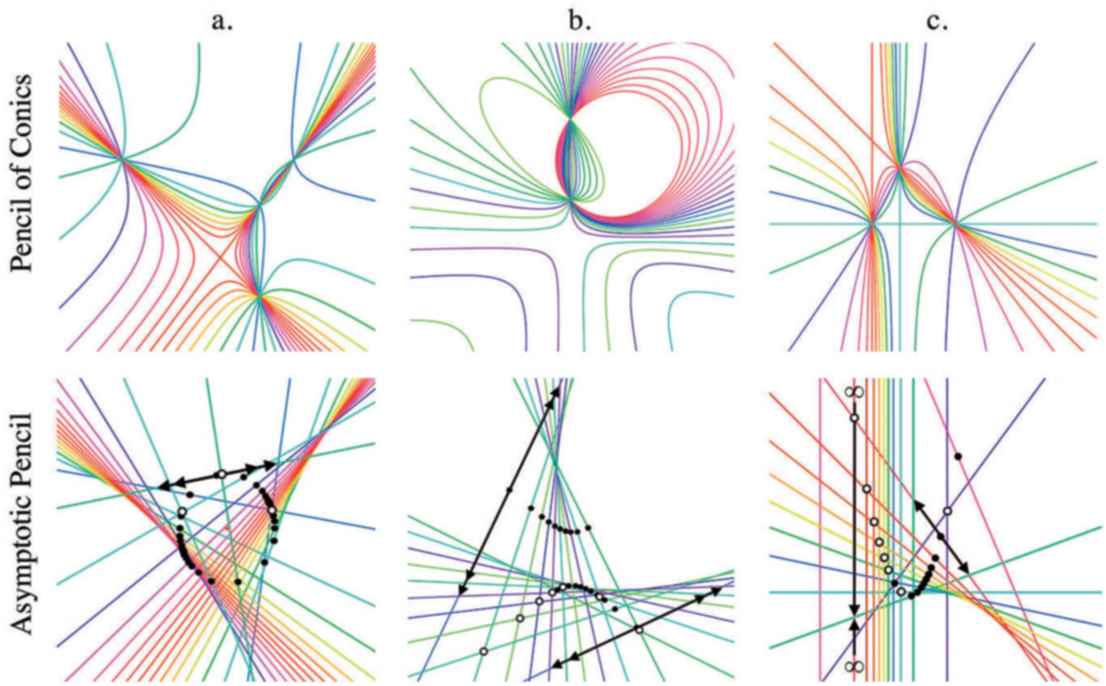

FIGURE 1. Selected conics from three pencils of conics (top row) and their asymptotic pencils (bottom row). The asymptotes of each hyperbola in the top row appear in the asymptotic pencil below. By Theorem 6.3, the asymptotic pencils are bisector fields. The black points in the bisector fields are the midpoints of the bisectors; each white point is an intersection of the lines in a bisector pair, which is the center of the degenerate hyperbola. The black arrows in the bottom row help illustrate the bisection property

of bisectors paired with respect to the orthogonality relation not only bisects the quadrilateral but is in turn bisected by every bisector of the quadrilateral. Moreover, the midpoints of the bisectors lie on a conic, which turns out to be the nine-point conic of the quadrilateral, a conic that passes through nine distinguished points of the complete quadrilateral, and which, as discussed in [7], was studied by Steiner and Beltrami in the mid-nineteenth century and then rediscovered by British geometers several decades later. Thus bisector fields help complete a picture of the nine-point conic by giving an interpretation of the other points on the nine-point conic.

(2) In [6], we describe the dual curve of bisectors and use this to classify bisector fields in terms of envelopes of tangent lines. Over the field of real numbers or complex numbers, we obtain a complete classification of all bisector fields by finding the envelope of a bisector field. Beltrami's rational tricuspidal quartic thus reappears as one possible envelope. Over a finite field, the classification is more complicated and incomplete. In any case, bisector fields give a

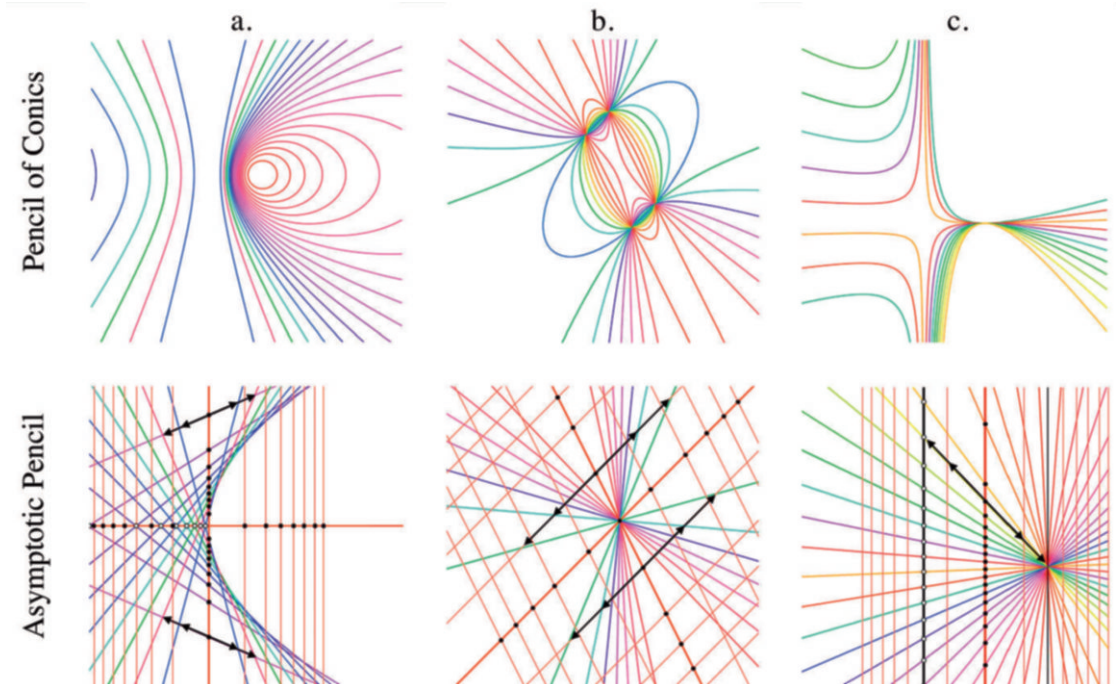

Figure 2. As in Fig. 1, the top row contains pencils of conics and the bottom row the corresponding asymptotic pencils and bisector fields. These examples differ from Fig. 1 in that each pencil contains reducible conics that are pairs of parallel lines. Each asymptotic pencil that contains a pair of parallel lines contains all pairs of parallel lines having the same midline, a property that is visible in the examples in the second row. The midline itself is a double line and contains the midpoints of all bisectors not parallel to it

new interpretation of the tangent lines of the Steiner deltoid, a curve and its tangent lines that appear in numerous contexts.

(3) Bisector fields can be viewed as equivalence classes of quadrilaterals or quadrangles. Call two quadrilaterals entangled if they have the same bisector field. In a future work we show entanglement reduces to a simple, *a priori* asymmetric, criterion, namely that the sides of one quadrilateral bisect the sides of the other with respect to the same midpoints. A quadrilateral is entangled with another quadrilateral if and only if the pencils of conics their vertices define are entangled also, in the sense that each hyperbola in one pencil shares asymptotes with a hyperbola in the other pencil. We use these equivalence classes to study moduli spaces of quadrilaterals.

(4) For $\mathbb{B}$ a collection of pairs of lines in the plane and p a point, let $\mathbb{B}(p)$ denote the set of points q for which q and the reflection of q through p lie on a pair in $\mathbb{B}$. In a forthcoming work, we show that if the underlying field $\Bbbk$ is sufficiently large (e.g., infinite), then $\mathbb{B}$ is a bisector arrangement if and

only if $\mathbb{B}(p)$ is a conic for each point p. Thus a collection of pairs of lines is a bisector arrangement if and only if the pairs are "a conic distance away" from each point of the plane. When $\Bbbk = \mathbb{R}$, there is an interesting and rather tight connection between these conics and the geometry of fields of ellipses used to describe the state of polarization of an electromagnetic wave in the theory of optics.

The outline of the paper is as follows. In Sect. 2 we establish some terminology for pencils in our affine setting, including the notion of a degeneration of a conic, a slight generalization of the affine concept of asymptotes of a hyperbola. Section 3 develops the notion of an asymptotic pencil, the set of degenerations of the conics in the pencil. It is shown in Corollary 3.6 that regardless of the field, every asymptotic pencil contains a degenerate hyperbola (and at least two if the field has more than 3 elements). When $\Bbbk$ is an algebraically closed field, the asymptotic pencil can be viewed as a planar cubic in five-dimensional projective space (Proposition 3.8).

A typical way a pencil of affine conics arises is as the set of conics through four points in general position. These four points are the vertices of a quadrilateral, and in Sect. 4 quadrilaterals are used to help describe technical properties of the asymptotic pencil, such as whether two different degenerate hyperbolas in the pencil can share a line (see Lemma 4.4). These properties are needed in proofs in the next sections.

Bisectors are introduced in Sect. 5, first as bisectors of collections of conics, then as bisectors of pairs of lines (i.e., degenerate conics) and bisectors of quadrilaterals. The main theorem of the section, Theorem 5.3, implies that a bisector of a pair of conics bisects all the pairs of asymptotes of the hyperbolas in the pencil generated by the conics, as well as all hyperbolas that share their asymptotes with hyperbolas in the pencil. Aspects of this theorem are closely connected with Desargues' Involution Theorem, and in Corollary 5.6 we derive this theorem as a corollary. To conclude the article, Sect. 6 contains the proof of the main theorem, Theorem 6.3, which shows that bisector fields and asymptotic pencils are the same thing in different guises.

It follows from Proposition 4.5 and Theorem 6.3 that the lines in a nontrivial asymptotic pencil are the bisectors of a quadrilateral, and this idea is a starting point for the sequel [5]. This notion of a bisector of a quadrilateral is in distinction to the area and perimeter bisectors of Berele and Catoiu [2,3], but interestingly there appears in their context also an analytic analogue of the tricuspidal envelope of bisectors that was discussed above and is studied in [6].

We used Maple to generate the figures in the article.

2. Preliminaries

Since we work with arbitrary fields and outside the typical projective setting for pencils and conics, we first establish terminology. Throughout the paper $\Bbbk$ denotes a field of characteristic other than 2. An *affine conic* is the zero locus in $\Bbbk^2$ of a quadratic polynomial in $\Bbbk[X,Y]$ (quadratic, for short). Since we work with affine conics only, we will usually drop the adjective "affine." A quadratic polynomial f need not be determined by its zero set, but even if $\Bbbk$ is finite, if there is more than one zero of f, then the zero set uniquely determines f up to scalar multiple among quadratics [4, 16.1.4, p. 171]. An affine conic is *reducible* if it is the union of two lines over $\Bbbk$; equivalently, the quadratic that defines the conic is a product of linear polynomials.

A *hyperbola* over the field $\Bbbk$ is a conic that has two distinct points at infinity. A hyperbola has a *center*, a point of symmetry for the zero set of the hyperbola. The *asymptotes* of a hyperbola are the two lines through the center of the hyperbola that meet the hyperbola at infinity.

A *parabola* is a conic with one point at infinity and an *ellipse* a conic with no points at infinity. A pair of parallel lines or a double line is thus a degenerate parabola.

For the sake of expediency, in this article we say two quadratics f_1 and f_2 are *independent* if $\deg(\alpha f_1 + \beta f_2) = 2$ for all $\alpha, \beta \in \Bbbk$ where α and β are not both 0. Thus two quadratics are independent if and only if their degree 2 homogeneous parts are not scalar multiples of each other; if and only if over the algebraic closure of $\Bbbk$, the zero sets of the two quadratics do not share the same points at infinity. A *pencil of affine conics* is a set of conics that are the zero sets of the nonzero polynomials in $\Bbbk f_1 + \Bbbk f_2 = \{\alpha f_1 + \beta f_2 : \alpha, \beta \in \Bbbk\}$ for some independent quadratics f_1, f_2. (Since f_1 and f_2 are independent, all the nonzero polynomials in $\Bbbk f_1 + \Bbbk f_2$ are quadratics.) Thus a pencil of affine conics is parameterized by the projective line over $\Bbbk$.

If two conics in a pencil intersect in a point then all conics in the pencil intersect in this point. Moreover, if as in Figs. 1a and 2b two conics in the pencil intersect in four distinct points, no three of which are collinear, then the pencil is the set of conics through these four points, the *basepoints* of the pencil.

We need a slight generalization of asymptotes:

Definition 2.1. A *degeneration of a quadratic* f is a quadratic g that is reducible over $\Bbbk$ and for which $f - g \in \Bbbk$. A *degeneration of an affine conic* is the zero set of a degeneration of the defining quadratic of the conic.

Thus a degeneration of a conic is a pair of lines, possibly parallel and possibly a double line. Every conic has a degeneration over the algebraic closure of $\Bbbk$, but this need not be the case over $\Bbbk$ itself, as is evident from the next proposition.

Proposition 2.2. *A hyperbola has a unique degeneration, namely its asymptotes. A degenerate parabola, i.e., a pair of parallel lines ℓ, ℓ', has as its degenerations the pairs of lines parallel to ℓ and ℓ' that share the same midline as the pair ℓ, ℓ'. Neither an ellipse nor a nondegenerate parabola has a degeneration.*

Proof. Suppose a conic has a degeneration into a pair of lines defined by linear polynomials g_1 and g_2, and suppose $g_1g_2 - h_1h_2 \in \Bbbk$, where the h_i are linear polynomials in $\Bbbk[X, Y]$. If g_1 and g_2 define lines that are not parallel, then after a change of variables we can assume $g_1(X, Y) = X$ and $g_2(X, Y) = Y$. Since $XY + \lambda$ is irreducible for all $\lambda \neq 0$, the only degeneration of g_1g_2 is itself. It follows that the pair of asymptotes of a hyperbola is the unique degeneration of the hyperbola.

If instead g_1 and g_2 define parallel lines, then we can assume $g_1(X, Y) = X$ and $g_2(X, Y) = X - a$ for some $a \in \Bbbk$. The degenerations of g_1g_2 are conics of the form $(X - r)(X - s)$, where $r, s \in \Bbbk$ and $a = r + s$. The midline of the two lines $X = r$ and $X = s$ is $X = a/2$, which is the midline of the lines g_1 and g_2. This proves the second assertion of the proposition.

That an ellipse does not have a degeneration is clear since a conic and its degeneration share the same points at infinity and an ellipse does not meet the line at infinity. To see that a nondegenerate parabola f does not have a degeneration, use an affine transformation to reduce to the case that $f(X, Y) = Y - X^2$. Then $f + \lambda$ is an irreducible polynomial for all $\lambda \in \Bbbk$, and so f does not have a degeneration. □

3. Asymptotic pencils

As noted in the last section, we will typically drop the "affine" in "affine conic" since we focus in this article only on the affine case. Our main object of interest is that of an asymptotic pencil.

Definition 3.1. A set $\mathbb{A}$ of reducible conics (i.e., pairs of lines) is an *asymptotic pencil* if there is a pencil of conics for which $\mathbb{A}$ consists of the pairs of asymptotes of the hyperbolas in the pencil and the pairs of parallel lines that share a midline with a pair of parallel lines in the pencil, if any such pairs exist.

The next proposition, which is an immediate consequence of Proposition 2.2, gives a simple interpretation of an asymptotic pencil that will be useful in proofs.

Proposition 3.2. *A collection of pairs of lines is an asymptotic pencil if and only if it is the set of degenerations of conics in a pencil of conics.*

Figures 1 and 2 give examples of asymptotic pencils. By definition, independent quadratics generate pencils of conics, and the next lemma shows that any choice of two independent quadratics in the pencil will do.

Lemma 3.3. *Let f_1, f_2 be independent quadratics. Any two pairs of independent quadratics in $\Bbbk f_1 + \Bbbk f_2$ generate the same pencil, and any two pairs of independent quadratics in $\Bbbk f_1 + \Bbbk f_2 + \Bbbk$ generate the same asymptotic pencil.*

Proof. The argument is routine. Let g_1, g_2 be independent quadratics in $\Bbbk f_1 + \Bbbk f_2 + \Bbbk$. There are $a_{ij} \in \Bbbk$ such that $g_i = a_{i1} f_1 + a_{i2} f_2 + a_{i3}$ for $i = 1, 2$. If $a_{11}a_{22} - a_{12}a_{21} = 0$, then there are $\alpha, \beta \in \Bbbk$, not both 0, such that $\alpha a_{11} = \beta a_{21}$ and $\alpha a_{21} = \beta a_{22}$, and hence $\alpha g_1 - \alpha a_{31} = \beta g_2 - \beta a_{31}$, contradicting the independence of the pair g_1, g_2. Thus $a_{11}a_{22} - a_{12}a_{21} \neq 0$, from which it follows that there are $b_{ij} \in \Bbbk$ such that $f_i = b_{i1} g_1 + b_{i2} g_2 + b_{i3}$ for $i = 1, 2$. This implies $\Bbbk f_1 + \Bbbk f_2 + \Bbbk = \Bbbk g_1 + \Bbbk g_2 + \Bbbk$, and hence the asymptotic pencil generated by f_1, f_2 is the same at that generated by g_1, g_2. Choosing instead g_1 and g_2 in $\Bbbk f_1 + \Bbbk f_2$ and $a_{13} = a_{23} = 0$ yields $\Bbbk f_1 + \Bbbk f_2 = \Bbbk g_1 + \Bbbk g_2$. □

The next lemma implies that any two distinct conics in a pencil of affine conics are defined by independent quadratics.

Lemma 3.4. *Let f_1, f_2 be independent quadratics.*

(1) *Two quadratics $g_1, g_2 \in \Bbbk f_1 + \Bbbk f_2$ are dependent if and only if one of g_1, g_2 is a scalar multiple of the other.*
(2) *Two reducible quadratics in $\Bbbk f_1 + \Bbbk f_2 + \Bbbk$ are dependent if and only if they define the same pair of non-parallel lines or they define pairs of parallel lines having the same midline.*

Proof. For statement (1), write $g_i = \alpha_i f_1 + \beta_i f_2$, where $\alpha_i, \beta_i \in \Bbbk$ are not both 0. Suppose g_1 and g_2 are dependent. Then there are $\alpha, \beta \in \Bbbk$, not both 0, such that $\deg(\alpha g_1 + \beta g_2) < 2$. Since f_1 and f_2 are independent and

$$\alpha g_1 + \beta g_2 = (\alpha\alpha_1 + \beta\alpha_2) f_1 + (\alpha\beta_1 + \beta\beta_2) f_2,$$

it follows that $\alpha\alpha_1 + \beta\alpha_2 = \alpha\beta_1 + \beta\beta_2 = 0$. Since at least one of α and β is not zero, $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$, and hence the vectors (α_1, β_1) and (α_2, β_2) in $\Bbbk^2$ are linearly dependent. Consequently, g_1 and g_2 are scalar multiples of each other. The converse of (1) is clear.

To verify (2), suppose g_1 and g_2 are reducible quadratics in $\Bbbk f_1 + \Bbbk f_2 + \Bbbk$. There are $\lambda_1, \lambda_2 \in \Bbbk$ such that $g_1 + \lambda_1, g_2 + \lambda_2 \in \Bbbk f_1 + \Bbbk f_2$. Assume g_1 and g_2 are dependent. Then so are $g_1 + \lambda_1$ and $g_2 + \lambda_2$, and by (1), there is $0 \neq \gamma \in \Bbbk$ such that $\gamma(g_1 + \lambda_1) = g_2 + \lambda_2$, so that γg_1 is a degeneration of g_2. By Proposition 2.2, no degenerate hyperbola is the degeneration of a different degenerate hyperbola, so if g_1 and g_2 are hyperbolas, then g_1 and g_2 define the same pairs of non-parallel lines. If instead g_1 and g_2 define pairs of parallel lines, then by Proposition 2.2, as degenerations of each other, g_1 and g_2 must have zero sets that are pairs of parallel lines that share the same midline. The converse of (2) also follows from Proposition 2.2. □

In order to show that asymptotic pencils are nonempty, it suffices to prove that pencils of affine conics contain at least one hyperbola.

Proposition 3.5. *Every pencil of affine conics contains at least one hyperbola, and if* $|\Bbbk| > 3$ *the pencil contains at least two hyperbolas.*

Proof. Let f_1, f_2 be independent quadratics. For each $i = 1, 2$, write

$$f_i(X,Y) = a_iX^2 + b_iXY + c_iY^2 + d_iX + e_iY + g_i,$$

where the coefficients of f_i are from $\Bbbk$. Since f_1 and f_2 are independent, the vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) in $\Bbbk^3$ are linearly independent. The matrix having these two vectors as its rows has rank 2, and so its reduced row echelon form is one of the following matrices, where $b, c, c' \in \Bbbk$.

$$\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & c' \end{bmatrix} \quad \begin{bmatrix} 1 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By switching f_1 with f_2 and/or interchanging the variables X and Y, the only cases that need to be considered are the first matrix or the second matrix with $b = 0$. The latter case allows a reduction to

$$f_1(X,Y) = X^2 + d_1X + e_1Y + g_1$$
$$f_2(X,Y) = Y^2 + d_2X + e_2Y + g_1,$$

and so $f_1 - f_2$ is a hyperbola in $\Bbbk f_1 + \Bbbk f_2$ (it has two points at infinity) and, if $|\Bbbk| > 3$, then we can choose $t \in \Bbbk$ such that $t^2 \neq 1$, so that $f_1 - t^2 f_2$ is a hyperbola in $\Bbbk f_1 + \Bbbk f_2$ that is independent from $f_1 - f_2$.

In the former case, that of the first matrix in reduced row echelon form, we can reduce to the situation in which the quadratics f_1 and f_2 are

$$f_1(X,Y) = X^2 + c_1Y^2 + d_1X + e_1Y + g_1$$
$$f_2(X,Y) = XY + c_2Y^2 + d_2X + e_2Y + g_2.$$

First observe that f_2 is a hyperbola with points at infinity $[-c_2 : 1 : 0]$ and $[1 : 0 : 0]$, so the pencil $\Bbbk f_1 + \Bbbk f_2$ contains at least one hyperbola.

Now suppose $|\Bbbk| > 3$. There is $0 \neq r \in \Bbbk$ such that $r \neq -c_2$ and $r^2 + 2c_2r - c_1 \neq 0$. Let

$$\beta = -\frac{r^2 + c_1}{c_2 + r} \quad \text{and} \quad s = -\beta - r.$$

A calculation shows the degree two homogeneous component of $f_1 + \beta f_2$ is

$$X^2 + c_1Y^2 + \beta(XY + c_2Y^2) = (X - rY)(X - sY)$$

Also, $r \neq s$ since by the choice of r,

$$r - s = 2r + \beta = 2r - \frac{r^2 + c_1}{c_2 + r} = \frac{r^2 + 2c_2r - c_1}{r + c_2} \neq 0.$$

Therefore, $f_1 + \beta f_2$ is a hyperbola whose distinct points at infinity are $[r : 1 : 0]$ and $[s : 1 : 0]$, and so $f_1 + \beta f_2$ and f_2 are independent hyperbolas in $\Bbbk f_1 + \Bbbk f_2$ since they do not share the same points at infinity. □

Corollary 3.6. *Every asymptotic pencil contains a degenerate hyperbola, and if* $|\mathbb{k}| > 3$ *every asymptotic pencil contains at least two degenerate hyperbolas.*

Proof. By Proposition 3.5, every pencil of affine conics contains a hyperbola, and hence every asymptotic pencil contains a degenerate hyperbola. Suppose $|\mathbb{k}| > 3$. By Proposition 3.5, each pencil of affine conics contains at least two independent hyperbolas. If the asymptotes of these two hyperbolas are the same pairs of lines, then by Proposition 2.2, these asymptotes are degenerations of the hyperbolas, and so the equations for the hyperbolas differ by a constant, a contradiction to the fact that the hyperbolas are independent. Thus the asymptotic pencil contains at least two degenerate hyperbolas. □

Proposition 3.5 and Corollary 3.6 are not true if $\mathbb{k}$ has only 3 elements:

Example 3.7. Let $\mathbb{k}$ be the field with 3 elements. The pencil generated by the parabola $X^2 + Y$ and the degenerate hyperbola $XY + Y^2$ has, up to scalar multiple, only two other conics, the parabola $X^2+Y+XY+Y^2 = (Y+2X)^2+Y$ and the ellipse $X^2+Y+2XY+2Y^2$, whose zero set consists of the four points $(0, 0), (0, 1), (1, 1), (1, 2)$. By Proposition 2.2, the asymptotic pencil of $X^2 + Y$ and $XY + Y^2$ contains only one conic up to scalar multiple, the degenerate hyperbola $XY + Y^2$.

Viewing two polynomials in $\mathbb{k}[X, Y]$ as equivalent if one is a nonzero scalar multiple of the other, the space of equivalence classes of polynomials of degree at most 2 is five-dimensional projective space $\mathbb{P}^5(\mathbb{k})$, where an equivalence class of a polynomial $aX^2 + bXY + cY^2 + dX + eY + f$ is represented by a point $[a : b : c : d : e : f] \in \mathbb{P}^5(\mathbb{k})$. Pencils of affine conics correspond to lines under this representation. Asymptotic pencils are more complicated.

Proposition 3.8. *If* $\mathbb{k}$ *is an algebraically closed field, then an asymptotic pencil is a planar cubic in* $\mathbb{P}^5(\mathbb{k})$ *with one point missing.*

Proof. Let f_1, f_2 be independent quadratics. We show the asymptotic pencil generated by f_1, f_2 is a planar cubic. For each $i = 1, 2$, write

$$f_i(X, Y) = a_iX^2 + 2b_iXY + c_iY^2 + 2d_iX + 2e_iY + g_i,$$

where the coefficients of f_i are from $\mathbb{k}$. Let $\Delta(T, U, V)$ be the determinant of the following matrix, where T, U, V are indeterminates:

$$\begin{bmatrix} a_1U + a_2V & b_1U + b_2V & d_1U + d_2V \\ b_1U + b_2V & c_1U + c_2V & e_1U + e_2V \\ d_1U + d_2V & e_1U + e_2V & g_1U + g_2V + T \end{bmatrix}.$$

Let $\lambda, \alpha, \beta \in \mathbb{k}$, not all zero. Since $\mathbb{k}$ is algebraically closed, the quadratic $\alpha f_1 + \beta f_2 + \lambda$ is reducible over $\mathbb{k}$ if and only if $\Delta(\lambda, \alpha, \beta) = 0$. Viewing Δ as a polynomial over the ring $\mathbb{k}[U, V]$, the coefficient of T is

$$\Phi(U, V) = (a_1c_1 - b_1^2)U^2 + (a_1c_2 + a_2c_1 - 2b_1b_2)UV + (a_2c_2 - b_2^2)V^2.$$

Since the degree of T in Δ is at most 1, any choice of point $[\alpha : \beta]$ on the projective line $\mathbb{P}^1(\Bbbk)$ over $\Bbbk$ that is not a zero of Φ results in a choice for T that is a root of $\Delta(T, \alpha, \beta)$, and hence yields a reducible quadratic in the asymptotic pencil generated by f_1, f_2. Since $\Bbbk$ is algebraically closed, verifying the existence of such a point $[\alpha : \beta]$ amounts to showing Φ is not uniformly zero.

We claim the polynomial Φ is not uniformly zero. Suppose to the contrary that it is. For each i, since f_i is a quadratic, it cannot be that all of a_i, b_i, c_i are zero. Suppose $a_1 = 0$. From the coefficient of U^2 in Φ we obtain $b_1 = 0$ and hence $c_1 \neq 0$. Examination of the coefficient of UV shows then that $a_2 = 0$, and hence, using the coefficient of V^2, $b_2 = 0$, in which case $c_2 \neq 0$. In summary, $a_1 = b_1 = a_2 = b_2 = 0$, $c_1 \neq 0$ and $c_2 \neq 0$. Thus, with $\alpha = c_2$ and $\beta = -c_1$, the polynomial $\alpha f_1 + \beta f_2$ has degree at most 1, contrary to the fact that f_1 and f_2 are independent. This contradiction implies that if Φ is uniformly 0, then $a_1 \neq 0$. A symmetrical argument shows $a_2 \neq 0$.

Still assuming that Φ is uniformly 0, the fact that $a_1 \neq 0$ and $a_2 \neq 0$ implies $c_1 = b_1^2/a_1$ and $c_2 = b_2^2/a_2$. Substituting this into the coefficient of UV, we conclude $a_1b_2 - a_2b_1 = 0$. Since a_1 and a_2 are nonzero, the elements b_1 and b_2 are nonzero also. From the coefficients of U^2 and V^2, we obtain $b_1/a_1 = c_1/b_1$ and $b_2/a_2 = c_2/b_2$. From this and the fact that $a_1b_2 - a_2b_1 = 0$ it follows that $c_1/b_1 = c_2/b_2$, which implies $b_1c_2 - c_1b_2 = 0$. Thus $a_1b_2 - a_2b_1 = b_1c_2 - c_1b_2 = 0$, and so the vectors $(a_1, 2b_1, c_1)$ and $(a_2, 2b_2, c_2)$ in $\Bbbk^3$ are linearly dependent. This implies there are $\alpha, \beta \in \Bbbk$, not both 0, for which $\alpha f_1 + \beta f_2$ has degree at most 1, a contradiction that implies Φ is not uniformly 0.

The asymptotic pencil generated by f_1 and f_2 is the set of quadratics $\alpha f_1 + \beta f_2 + \lambda$ such that $[\alpha : \beta : \lambda] \in \mathbb{P}^2(\Bbbk)$, $\Delta(\lambda, \alpha, \beta) = 0$ and α and β are not both 0. The polynomial Δ is a cubic since the polynomial Φ is not uniformly zero. The dual points of the quadratics in $\Bbbk f_1 + \Bbbk f_2 + \Bbbk$ lie in a plane in $\mathbb{P}^5(\Bbbk)$, and so the asymptotic pencil generated by f_1 and f_2 is a cubic in this plane, minus the conic corresponding to $[0 : 0 : 1]$, i.e., the (empty) affine conic that is defined by the constant polynomial 1. □

If $\Bbbk$ is not an algebraically closed field, then by extending $\Bbbk$ to its algebraic closure, it follows that an asymptotic pencil over $\Bbbk$ is a piece of the cubic from Proposition 3.8 over the algebraic closure of $\Bbbk$. The extra points on the cubic that are not on the asymptotic pencil over $\Bbbk$ are the dual points of the ellipses in the original pencil of quadratics.

4. Pencils generated by quadrilaterals

Classically, pencils of conics arise in connection with vertices of complete quadrilaterals (or quadrangles) in the projective plane because of the fact that the set of projective conics through any four distinct points in general position is a pencil; see Figs. 1c and 2b. However, there are pencils that are not defined by the vertices of quadrilaterals in the affine plane, as in Figs. 1a, b, 2a, c, and so there are more pencils than quadrilaterals. By contrast, every nontrivial asymptotic pencil—nontrivial in the sense defined below—can be viewed as given by a quadrilateral. The main purpose of this section is to prove this assertion.

By a (*complete*) *quadrilateral* $Q = ABA'B'$ we mean two pairs of lines A, A' and B, B' in the affine plane such that neither pair is a translation of the other, these pairs do not share a line, the lines A, A', B, B' do not all share a point, and these four lines are not all parallel. The pairs A, A' and B, B' are the pairs of *opposite sides* of Q; the other pairs are *adjacent* sides of Q. The intersection of two adjacent sides of Q is a *vertex*. (If a pair of adjacent sides has parallel lines, then the vertex is the point at infinity for these sides.) The lines joining the pairs of opposite vertices are the *diagonals* of Q. The definition implies a quadrilateral Q has more than one vertex, that no adjacent sides are equal and that the line at infinity is neither a diagonal or a side of Q. Opposite sides can be equal, in which case Q is a *degenerate* quadrilateral.

Definition 4.1. The *pencil of* Q is the pencil defined by the reducible quadratics whose zero sets are the pairs of opposites sides of Q.

Proposition 4.5 will show that as long as an asymptotic pencil is not trivial, in the following sense, there is a quadrilateral that generates it.

Definition 4.2. An asymptotic pencil is *trivial* if it consists only of degenerate hyperbolas, all of which have the same center.

Proposition 4.3. *If* $\Bbbk$ *is an algebraically closed field, then no asymptotic pencil is trivial.*

Proof. We prove first that if f_1 and f_2 are independent quadratics defining degenerate hyperbolas sharing the same center, then the pencil generated by f_1 and f_2 contains a double line. Since the zero sets of f_1 and f_2 each have two points at infinity, we can assume after an affine transformation that

$$f_1(X,Y) = XY \text{ and } f_2(X,Y) = (aX+bY)(cX+dY), \text{ where } a,b,c,d \in \Bbbk.$$

Since f_2 is not a double line and f_1 and f_2 are independent, all of the following conditions must hold:

$$ad - bc \neq 0 \quad a \neq 0 \text{ or } d \neq 0 \quad b \neq 0 \text{ or } c \neq 0 \quad a \neq 0 \text{ or } b \neq 0 \quad c \neq 0 \text{ or } d \neq 0. \tag{1}$$

To prove there is a double line in the pencil generated by f_1 and f_2, we show there are pairs α, β and e, f in $\Bbbk$ such that at least one member of each pair is nonzero and

$$\alpha XY + \beta(aX+bY)(cX+dY) = (eX+fY)^2. \tag{2}$$

Expanding and collecting like terms, it follows that such a double line $(eX + fY)^2$ can be found in the pencil if and only if

$$\beta ac - e^2 = 0 \quad \alpha + \beta(ad + bc) - 2ef = 0 \quad \beta bd - f^2 = 0.$$

If $b = 0$, then from (1) we have $a \neq 0$, $c \neq 0$ and $d \neq 0$, and so with

$$\alpha = -\frac{ad + bc}{ac} \quad \beta = \frac{1}{ac} \quad e = 1 \quad f = 0$$

we obtain Eq. (2). Similarly, if $d = 0$, we may choose α, β, e, f to obtain Eq. (2).

Now suppose $b \neq 0$ and $d \neq 0$. Since $\Bbbk$ is algebraically closed, there is $\theta \in \Bbbk$ such that $bd\theta^2 = ac$. With the assignments

$$\alpha = \frac{2bd\theta - ad - bc}{bd} \quad \beta = \frac{1}{bd} \quad e = \theta \quad f = 1$$

we obtain Eq. (2). Therefore, in all cases the asymptotic pencil generated by f_1 and f_2 contains a double line.

The proposition now follows since Corollary 3.6 and the fact that $\Bbbk$ is infinite imply there are at least two independent degenerate hyperbolas in any asymptotic pencil over $\Bbbk$. If these two hyperbolas do not have the same center, then the asymptotic pencil is nontrivial. If they do have the same center, then, by what we have established, the pencil they generate contains a double line, and so the asymptotic pencil is not trivial since it contains this double line also. □

It is easy to construct examples of trivial asymptotic pencils over non-algebraically closed fields. For example, if $\Bbbk$ is the field of real numbers, the asymptotic pencil for the conics XY and $X^2 - Y^2$ is trivial, as can be checked directly or using the calculations in the proof of Proposition 4.3. A more general explanation for this example is that the two degenerate hyperbolas that define the asymptotic pencil are interleaved, with each linear component of each hyperbola between the two linear components of the other hyperbola. Whenever this is the case, the two degenerate hyperbolas sharing the same center will define a trivial asymptotic pencil. We will be interested in nontrivial asymptotic pencils, there being little to say about the trivial case.

The asymptotic pencil in Fig. 2c illustrates the next lemma. In that figure, the black line is shared by every hyperbola in the asymptotic pencil.

Lemma 4.4. *The following are equivalent for an asymptotic pencil* $\mathbb{A}$.

(1) $\mathbb{A}$ *contains two degenerate hyperbolas that intersect in a line.*
(2) $\mathbb{A}$ *contains a degenerate parabola and a degenerate hyperbola that intersect in a line.*
(3) $\mathbb{A}$ *contains two conics that intersect in a line.*

In this case, all hyperbolas in $\mathbb{A}$ *intersect in the same line.*

Proof. We will prove the last assertion of the lemma in the course of proving (1) implies (2).

(1) $\Rightarrow$ (2): Suppose g_1 and g_2 define degenerate hyperbolas in $\mathbb{A}$ that share a line. After an affine transformation, we may assume $g_1(X,Y) = XY$ and the shared line is $X = 0$. Write $g_2(X,Y) = X(dX + eY + f)$, where $d, e, f \in \Bbbk$ and d and e are not both 0. Since g_2 is a hyperbola, $e \neq 0$. A quadratic in $\Bbbk g_1 + \Bbbk g_2 + \Bbbk$ has the form

$$\alpha XY + \beta X(dX + eY + f) + \lambda = X(\beta dX + (\beta e + \alpha)Y + \beta f) + \lambda,$$

where $\alpha, \beta, \lambda \in \Bbbk$ and α, β are not both 0. By Proposition 2.2, a necessary condition for such a quadratic to be reducible is that $\lambda = 0$ or $\beta e + \alpha = 0$. If this quadratic is a hyperbola, then $\beta e + \alpha \neq 0$ and $\lambda = 0$, and so we have the degenerate hyperbola $X(\beta dX + (\beta e + \alpha)Y + \beta f)$, which shares the linear component X with g_1 and g_2. Note also that $\mathbb{A}$ contains the degenerate parabola

$$-eXY + X(dX + eY + f) = X(dX + f),$$

and hence by Proposition 2.2 contains a degenerate parabola that shares a linear component with the hyperbola XY in $\mathbb{A}$. This proves that if $\mathbb{A}$ contains two independent hyperbolas that share a linear component, then $\mathbb{A}$ contains a degenerate parabola and a degenerate hyperbola that intersect in a line, and all hyperbolas in $\mathbb{A}$ intersect in this same line.

(2) $\Rightarrow$ (1): Suppose $\mathbb{A}$ contains a degenerate hyperbola $g_1 = 0$ and a degenerate parabola $g_2 = 0$ that intersect in the same line. Without loss of generality,

$$g_1(X,Y) = XY \text{ and } g_2(X,Y) = X(X - a) \text{ for some } a \in \Bbbk.$$

Thus $\mathbb{A}$ contains the hyperbola

$$XY + X(X - a) = X(X + Y - a).$$

This hyperbola and the hyperbola defined by g_1 are distinct and intersect in a line.

(2) $\Leftrightarrow$ (3): It is clear (2) implies (3). Conversely, suppose g_1 and g_2 are independent quadratics whose zero sets are in $\mathbb{A}$ and share a linear component. If g_1 and g_2 are both hyperbolas, then since we have shown already that (1) and (2) are equivalent, statement (2) follows. So suppose without loss of generality that g_1 is not a hyperbola. Then, since g_1 is reducible, g_1 is a parabola. If g_2 is also a parabola, then since g_1 and g_2 share a linear component, all the linear components of g_1 and g_2 are parallel, contrary to the assumption that g_1 and g_2 are independent. Thus g_2 is a hyperbola, and so $\mathbb{A}$ contains a hyperbola and a parabola that share a line. $\square$

By Corollary 3.6, if $|\Bbbk| > 3$, then every asymptotic pencil can be specified by two degenerate hyperbolas, in fact by any two different reducible quadratics in the asymptotic pencil. If these degenerate hyperbolas intersect in a line, they do not define a pair of opposite sides of a quadrilateral. However, as we show next, as long as the asymptotic pencil is nontrivial, these reducible conics can be traded for two more that make a quadrilateral. This is true even if $|\Bbbk| = 3$, but to obtain a non-degenerate quadrilateral we need once again that $|\Bbbk| > 3$.

Proposition 4.5. *An asymptotic pencil $\mathbb{A}$ is nontrivial if and only if $\mathbb{A}$ is the asymptotic pencil of some quadrilateral Q. If $|\Bbbk| > 3$, then Q can be chosen a non-degenerate quadrilateral.*

Proof. If $\mathbb{A}$ is the asymptotic pencil of a quadrilateral Q, the pairs of opposite sides of Q define reducible conics in $\mathbb{A}$ that, if they are hyperbolas, do not share the same center. This guarantees $\mathbb{A}$ is nontrivial. To prove the converse, first note that by Corollary 3.6 there is a degenerate hyperbola f_1 in $\mathbb{A}$. Since $\mathbb{A}$ is nontrivial, $\mathbb{A}$ contains either a hyperbola with a different center than that of f_1 or a degenerate parabola.

If $\mathbb{A}$ contains a degenerate parabola f_2, then by Proposition 2.2 and the fact that $|\Bbbk| \geq 3$, the asymptotic pencil $\mathbb{A}$ contains a double line and a degenerate parabola whose linear components are distinct and parallel to the double line. One of these two degenerate parabolas does not share a linear component with f_1 and hence forms a quadrilateral with the zero set of f_1. Thus if $\mathbb{A}$ contains a degenerate parabola, $\mathbb{A}$ contains a quadrilateral Q and $\mathbb{A}$ is the asymptotic pencil of Q.

If $\mathbb{A}$ does not contain a parabola, then $\mathbb{A}$ contains a hyperbola f_2 that does not share a center with f_1. Since $\mathbb{A}$ does not contain a parabola, Lemma 4.4 implies f_1 and f_2 do not share a linear component, and hence f_1 and f_2 define a quadrilateral Q. Lemma 3.4 implies f_1 and f_2 are independent, so $\mathbb{A}$ is the asymptotic pencil of Q. This proves the first assertion of the proposition.

Finally, suppose $|\Bbbk| > 3$. If Q is degenerate, then Q has a pair of parallel opposite sides that are equal. By Proposition 2.2 and the fact that $|\Bbbk| > 3$, $\mathbb{A}$ contains a double line and two degenerate parabolas whose zero sets are distinct and whose linear components are parallel to the double line. We can replace the double line that is a pair of opposite sides of Q with one of the degenerate parabolas that does not share a linear component with the other conic that defines Q. In doing so, we obtain a non-degenerate quadrilateral in $\mathbb{A}$. □

5. Bisectors of conics

For a line ℓ in the affine plane, denote by $\overline{\ell} = \ell \cup \{\infty\}$ the projective closure of ℓ, i.e., the line ℓ and its point at infinity ∞ in the projective plane. We define the *midpoint* of two points p, q on $\overline{\ell}$ as the usual midpoint of two points if $p, q \in \ell$, and as ∞ if $p \in \ell$ and $q = \infty$. In the first case the midpoint is *finite* and in the second it is *infinite*. If $p = q = \infty$, the midpoint is *undetermined*.

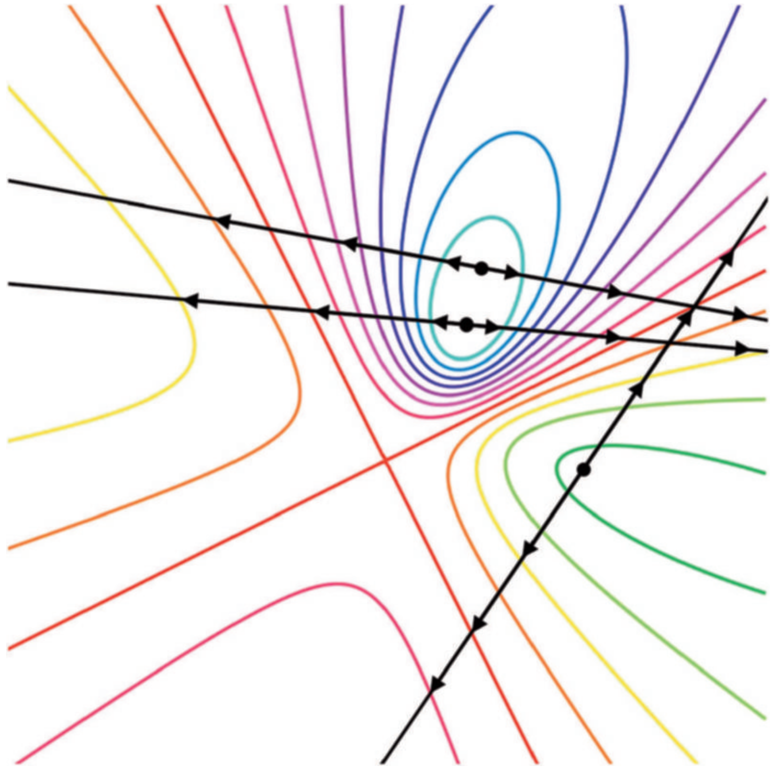

Figure 3. The three black lines are bisectors of the collection of conics in the figure. The midpoints are indicated by black dots and the arrows are to help visualize the bisection property

It will be convenient to have terminology for the different ways a line can intersect a conic. A line ℓ *meets* a conic defined by a quadratic f if there is a point, possibly at infinity, on both ℓ and f; i.e., the projective closure of ℓ has nonempty intersection with the projective closure of the zero set of f. The line ℓ *crosses* the zero set $V(f)$ (or f, for short) if ℓ is not a component of $V(f)$ and ℓ meets $V(f)$ in at least one point in the affine plane. Note that any line meets any reducible conic but may not cross it, since the line could be a component of the conic or meet the conic only at infinity. If ℓ crosses f, then $\mathrm{mid}_f(\ell)$ denotes the midpoint of the two crossing points, one of which may be at infinity.

Definition 5.1. A line ℓ *bisects* a collection $\mathscr{C}$ of conics (or the quadratics that define the conics) with *midpoint* m if $m = \mathrm{mid}_f(\ell)$ for all quadratics f whose zero sets are in $\mathscr{C}$ and are crossed by ℓ. If no conic in $\mathscr{C}$ is crossed by ℓ, then ℓ is vacuously a bisector of $\mathscr{C}$ and its midpoint $\mathrm{mid}_f(\ell)$ is *undetermined* for each quadratic f whose zero set is in $\mathscr{C}$.

Figure 3 illustrates the definition. The next lemma provides the first step in relating asymptotic pencils to the bisection property.

Lemma 5.2. *Let $\mathbb{A}$ be the asymptotic pencil generated by quadratics f_1 and f_2. A line ℓ that meets the zero sets of f_1 and f_2 bisects the pair $\{f_1, f_2\}$ if and only if ℓ is a component of a reducible conic in $\mathbb{A}$.*

Proof. After an affine transformation, we may assume ℓ is the line $Y = 0$. The line ℓ is a member of a pair in $\mathbb{A}$ if and only if there are $\alpha, \beta, \lambda, t, u, v \in \mathbb{k}$ such that α and β are not both 0; t and u are not both 0; and

$$\alpha f_1 + \beta f_2 = Y(tX - uY + v) + \lambda. \tag{3}$$

For each i, write

$$f_i(X, Y) = a_i X^2 + b_i XY + c_i Y^2 + d_i X + e_i Y + g_i.$$

Expanding, collecting like terms and equating coefficients, we obtain that Eq. (3) holds if and only if

$$t = \alpha b_1 + \beta b_2 \tag{4}$$
$$u = -\alpha c_1 - \beta c_2 \tag{5}$$
$$v = \alpha e_1 + \beta e_2 \tag{6}$$
$$\lambda = \alpha g_1 + \beta g_2 \tag{7}$$
$$0 = \alpha a_1 + \beta a_2 \tag{8}$$
$$0 = \alpha d_1 + \beta d_2. \tag{9}$$

Therefore, the line ℓ is a member of a pair in $\mathbb{A}$ if and only if there are $\alpha, \beta, \lambda, t, u, v \in \mathbb{k}$ such that α and β are not both 0; t and u are not both 0; and Eqs. (4)–(9) hold.

We prove the lemma by establishing a series of simple claims.

Claim 1: ℓ is a member of a pair in $\mathbb{A}$ if and only if $a_1 d_2 - a_2 d_1 = 0$.

There is not a solution to Eqs. (4)–(9) in which $t = u = 0$ but α and β are nonzero, since otherwise Eqs. (4), (5) and (8) with $t = u = 0$ imply $\alpha f_1 + \beta f_2$ is a polynomial of degree at most 1, contrary to the fact that f_1 and f_2 are independent. Using this observation and Eqs. (8) and (9), we conclude that ℓ is a member of a pair in $\mathbb{A}$ if and only if $a_1 d_2 - a_2 d_1 = 0$.

Claim 2: If $\mathrm{mid}_{f_1}(\ell)$ and $\mathrm{mid}_{f_2}(\ell)$ are both finite, then ℓ is a member of a pair in $\mathbb{A}$ if and only if ℓ bisects $\{f_1, f_2\}$.

Suppose $\mathrm{mid}_{f_1}(\ell)$ and $\mathrm{mid}_{f_2}(\ell)$ are both finite. Then ℓ crosses both f_1 and f_2 and does not meet either at infinity. Thus for each $i = 1, 2$, we have $a_i \neq 0$ since otherwise the line ℓ, which is given by $Y = 0$, meets f_i at a point at infinity. The points of intersection of ℓ with f_i are $(\sigma_i, 0)$ and $(\tau_i, 0)$, where σ_i and τ_i are the zeroes of $a_i X^2 + d_i X + g_i$. Necessarily, $\sigma_i + \tau_i = -d_i a_i^{-1}$. Therefore, $\sigma_1 + \tau_1 = \sigma_2 + \tau_2$ if and only if $d_1 a_2 - d_2 a_1 = 0$. The midpoint of the points where ℓ crosses f_i is $(\frac{1}{2}(\sigma_i + \tau_i), 0)$ and so ℓ bisects the pair $\{f_1, f_2\}$ if and only if $d_1 a_2 - d_2 a_1 = 0$. By Claim 1, this is the case if and only if ℓ is a member of a pair in $\mathbb{A}$.

Claim 3: If at least one of $\mathrm{mid}_{f_1}(\ell)$ or $\mathrm{mid}_{f_2}(\ell)$ is undetermined, then ℓ is a member of a pair in $\mathbb{A}$ and ℓ bisects $\{f_1, f_2\}$.

Without loss of generality, $\mathrm{mid}_{f_1}(\ell)$ is undetermined and so ℓ does not cross f_1. By assumption, ℓ meets f_1, so either ℓ is a component of f_1 or ℓ meets f_1 at infinity only. In either case, $a_1 = d_1 = 0$, so by Claim 1, ℓ is a member of a pair in $\mathbb{A}$, and by definition ℓ bisects $\{f_1, f_2\}$ since $\mathrm{mid}_{f_1}(\ell)$ is undetermined and hence the position of $\mathrm{mid}_{f_2}(\ell)$ does not matter for the sake of guaranteeing bisection of $\{f_1, f_2\}$.

Claim 4: If one of $\mathrm{mid}_{f_1}(\ell)$ and $\mathrm{mid}_{f_2}(\ell)$ is finite and the other is infinite, then ℓ is not a member of a pair in $\mathbb{A}$ and ℓ does not bisect $\{f_1, f_2\}$.

Since the two midpoints do not agree, ℓ does not bisect $\{f_1, f_2\}$. Without loss of generality, suppose $\mathrm{mid}_{f_1}(\ell)$ is infinite and $\mathrm{mid}_{f_2}(\ell)$ is finite. Then $a_1 = 0$ and $a_2 \neq 0$ since ℓ is the line $Y = 0$ and ℓ meets f_1 at infinity but does not meet f_2 at infinity. Thus $a_1 d_2 - d_1 a_2 = 0$ if and only if $d_1 = 0$. However, if $d_1 = 0$, then $\mathrm{mid}_{f_1}(\ell)$ is undetermined since also $a_1 = 0$, a contradiction. Thus Claim 1 implies ℓ is not a member of a pair in $\mathbb{A}$

Claim 5: If both $\mathrm{mid}_{f_1}(\ell)$ and $\mathrm{mid}_{f_2}(\ell)$ are infinite, then ℓ is a member of a pair in $\mathbb{A}$ and ℓ bisects $\{f_1, f_2\}$.

It is clear ℓ bisects $\{f_1, f_2\}$ since $\mathrm{mid}_{f_1}(\ell) = \mathrm{mid}_{f_2}(\ell)$. Also, since both midpoints are infinite, we have $a_1 = a_2 = 0$, which implies $a_1 d_2 - a_2 d_1 = 0$, so that ℓ is a member of a pair in $\mathbb{A}$ by Claim 1. □

The next theorem shows that bisection of a single pair of independent quadratics lifts to bisection of an entire "affine net" of these two conics. The first two paragraphs of the proof of the theorem can be avoided by using a general version of Desargues' Involution Theorem, such as in [4, 14.2.8.3, p. 125], but instead we use our bisector methods to prove the theorem and then in Corollary 5.6 derive Desargues' theorem as a consequence.

Theorem 5.3. *Let f_1 and f_2 be independent quadratics, and let ℓ be a line that meets the zero sets of f_1 and f_2. If ℓ bisects $\{f_1, f_2\}$, then ℓ bisects the set of quadratics in $\Bbbk f_1 + \Bbbk f_2 + \Bbbk$ and hence also the asymptotic pencil generated by f_1, f_2.*

Proof. Suppose ℓ bisects $\{f_1, f_2\}$. After an affine transformation, we may assume ℓ is the line $Y = 0$. Suppose first that $\mathrm{mid}_{f_1}(\ell)$ and $\mathrm{mid}_{f_2}(\ell)$ are undetermined. As in the proof of Claim 3 of Lemma 5.2, it follows that for $i = 1, 2$,

$$f_i(X, Y) = b_i XY + c_i Y^2 + e_i Y + h_i \text{ for some } b_i, c_i, e_i, h_i \in \Bbbk.$$

Every quadratic in $\Bbbk f_1 + \Bbbk f_2 + \Bbbk$ thus has this form, and so every such quadratic either has $Y = 0$ as a component or meets $Y = 0$ at infinity only. Thus ℓ does not cross any conic defined by a quadratric in $\Bbbk f_1 + \Bbbk f_2 + \Bbbk$, which implies ℓ vacuously bisects every conic defined by a quadratic in $\Bbbk f_1 + \Bbbk f_2 + \Bbbk$. For the rest of the proof we assume without loss of generality that ℓ crosses f_1 and so $\mathrm{mid}_{f_1}(\ell)$ is determined.

By Lemma 5.2, ℓ is a component of a conic in the asymptotic pencil generated by f_1, f_2. Let g be a quadratic in $\Bbbk f_1 + \Bbbk f_2 + \Bbbk$ that defines a conic ℓ crosses. Since $\mathrm{mid}_{f_1}(\ell)$ is determined, it suffices to show ℓ bisects the set of the three conics defined by f_1, f_2, g. If g and f_1 are independent, then the pairs f_1, f_2 and f_1, g generate the same asymptotic pencil by Lemma 3.3, and since ℓ meets g and f_1, and ℓ is a component of a conic in the asymptotic pencil defined by f_1, g, Lemma 5.2 implies that ℓ bisects the pair of conics defined by f_1 and g. Therefore, since $\mathrm{mid}_{f_1}(\ell)$ is determined, ℓ bisects g with midpoint $\mathrm{mid}_g(\ell) = \mathrm{mid}_{f_1}(\ell)$.

It remains to examine the case in which f_1 and g are dependent, and to prove in this case that $\mathrm{mid}_{f_1}(\ell) = \mathrm{mid}_g(\ell)$. Since $g \in \Bbbk f_1 + \Bbbk f_2 + \Bbbk$, there is $\lambda \in \Bbbk$ such that $g - \lambda \in \Bbbk f_1 + \Bbbk f_2$. Since $g - \lambda$ and f are dependent and in $\Bbbk f_1 + \Bbbk f_2$, Lemma 3.4(1) implies $g - \lambda = \gamma f_1$ for some $0 \neq \gamma \in \Bbbk$. Thus since $\mathrm{mid}_{f_1}(\ell) = \mathrm{mid}_{\gamma f_1}(\ell)$, it suffices to show that $\mathrm{mid}_g(\ell) = \mathrm{mid}_{g-\lambda}(\ell)$.

By assumption, ℓ crosses g, and since ℓ crosses f_1, ℓ also crosses $g-\lambda = \gamma f_1$. Write

$$g(X, Y) = aX^2 + bXY + cY^2 + dX + eY + h.$$

Since the line ℓ is given by $Y = 0$ and ℓ crosses g, the line ℓ is not a component of g and $aX^2 + dX + h$ has a root in $\Bbbk$. If $a = 0$, then ℓ meets g and $g - \lambda$ at infinity, as well as in the affine plane, and so $\mathrm{mid}_g(\ell) = \infty = \mathrm{mid}_{g-\lambda}(\ell)$. If $a \neq 0$, then since $aX^2 + dX + h$ has a root σ in $\Bbbk$, there is another root $\tau \in \Bbbk$. The points $(\sigma, 0)$ and $(\tau, 0)$ are where ℓ crosses g, and $\sigma + \tau = -d/a$. Similarly, $g - \lambda = aX^2 + dX + h - \lambda$ has roots $\sigma', \tau' \in \Bbbk$ with $\sigma' + \tau' = -d/a$. The line ℓ crosses $g - \lambda$ at $(\sigma', 0)$ and $(\tau', 0)$, so since $\sigma + \tau = \sigma' + \tau'$, we conclude $\mathrm{mid}_g(\ell) = -d/(2a) = \mathrm{mid}_{g-\lambda}(\ell)$, which proves the claim and completes the proof of the theorem. □

A line *bisects a quadrilateral* Q if it bisects the set of pairs of opposite sides of Q.

Corollary 5.4. *Let Q be a quadrilateral, and let f_1 and f_2 be quadratics that define the pairs of opposite sides of Q. A line bisects Q if and only if it bisects the set of conics defined by quadratics in $\Bbbk f_1 + \Bbbk f_2 + \Bbbk$.*

Proof. The quadratics f_1 and f_2 are independent since Q is a quadrilateral, so we apply Theorem 5.3. □

Corollary 5.5. *Let Q be a quadrilateral with four distinct vertices, one of which is possibly at infinity. If a line bisects Q, it bisects the set of conics through the vertices of Q.*

Proof. Since the set of conics through four distinct points in general position is a pencil, we apply Corollary 5.4. □

Corollary 5.6. (Desargues' Involution Theorem) *Let $\mathscr{P}$ be a pencil of conics in $\mathbb{P}^2(\Bbbk)$, and let ℓ be a line in $\mathbb{P}^2(\Bbbk)$ that does not pass through a basepoint of $\mathscr{P}$. There is an involution Λ on ℓ such that if p and q are the points of intersection of ℓ and a conic in $\mathscr{P}$, then p and q are conjugate under Λ.*

Proof. Let p_1, q_1 be the points of intersection of ℓ with a conic in $\mathscr{P}$ defined by quadratic f_1, and let p_2, q_2 be the points of intersection of ℓ with a conic in $\mathscr{P}$ defined by a quadratic f_2 such that $\{p_1, q_1\} \neq \{p_2, q_2\}$. There is a projective transformation T of the projective plane that carries ℓ onto a line $T(\ell)$ such that in some affine chart containing $T(p_1), T(q_1), T(p_2), T(q_2)$, the midpoint of $T(p_1)$ and $T(q_1)$ is the same as the midpoint of $T(p_2)$ and $T(q_2)$. In this chart, the dehomogenizations of f_1 and f_2 are independent by Lemma 3.4(1), and also in this chart, the line $f(\ell)$ bisects the images of f_1 and f_2 under T. By Corollary 5.4, $T(\ell)$ bisects all the conics through the vertices of Q. Let Λ be the reflection on $T(\ell)$ about the midpoint of $T(p_1)$ and $T(q_1)$ (which is also the midpoint of $T(p_2)$ and $T(q_2)$) that sends the point at infinity of $T(\ell)$ (with respect to the given chart) to itself. Then $T^{-1} \circ \Lambda \circ T$ is an involution on ℓ such that the pairs of points where the conics through the vertices of Q cross ℓ are conjugate. □

6. Asymptotic pencils and bisector fields

Since pairs of lines can be viewed as zero sets of reducible conics, the definition of bisection of conics in the last section subsumes that of bisection of collections of line pairs, and for such collections we single out in the next definition the property that "everything bisects everything."

Definition 6.1. A set $\mathbb{B}$ of paired lines is a *bisector arrangement* if each line ℓ in each pair in $\mathbb{B}$ bisects $\mathbb{B}$. We denote by $\mathrm{mid}_{\mathbb{B}}(\ell)$ the midpoint of ℓ as a bisector of $\mathbb{B}$. A bisector arrangement is *trivial* if either every pair is a translation of every other pair, all lines in the arrangement go through the same point, or all lines are parallel. A *bisector field* is a nontrivial bisector arrangement that cannot be extended to a larger bisector arrangement.

The trivial nature of a trivial bisector arrangement is that in these cases the bisector arrangement has the property that each line in the arrangement bisects every pair in the arrangement for uninteresting reasons: If all the lines in a collection of pairs of lines meet at a point p in the affine plane, then each line in the arrangement bisects each pair with midpoint p, while if all the lines in the pairs are parallel and hence meet at a point at infinity, then each line vacuously bisects each pair with undetermined midpoint. If instead, each pair is a translation of every other pair, then no matter the placement of the pairs in the collection, each line in the arrangement bisects the collection with midpoint at infinity or undetermined midpoint.

The next lemma, which is crucial for the main theorem of the article, can be viewed as a uniqueness statement for the pairings in a bisector arrangement. The bisector configuration in Fig. 2c shows the statement of the lemma is optimal.

Lemma 6.2. *Let $\mathbb{B} = \{\mathsf{P}_1, \mathsf{P}_2\}$ be a nontrivial bisector arrangement consisting of two line pairs P_1 and P_2. If P is a line pair such that $\mathbb{B} \cup \{\mathsf{P}\}$ is a bisector arrangement extending $\mathbb{B}$, then there is a line ℓ in P such that the only line pair P' containing ℓ for which $\mathbb{B} \cup \{\mathsf{P}'\}$ is a bisector arrangement extending $\mathbb{B}$ is $\mathsf{P}' = \mathsf{P}$.*

Proof. Let P be a line pair for which $\mathbb{B} \cup \{\mathsf{P}\}$ is a bisector arrangement. We will prove a series of claims about the statement

(∗) *There is a line ℓ in P such that the only line pair P' containing ℓ for which $\mathbb{B} \cup \{\mathsf{P}'\}$ is a bisector arrangement is $\mathsf{P}' = \mathsf{P}$.*

Write $\mathsf{P}_1 = \{A, A'\}$, $\mathsf{P}_2 = \{B, B'\}$ and $\mathsf{P} = \{C, C'\}$.

Claim 1. At most two of the lines in the arrangement $\mathbb{B}$ have an infinite midpoint.

Suppose without loss of generality that the bisectors A, A', B have infinite midpoints in $\mathbb{B}$. Then A and A' are each parallel to exactly one of B and B', and B is parallel to exactly one of A and A', so it follows that each line in each of the two pairs $\{A, A'\}$ and $\{B, B'\}$ is parallel to exactly one line in the other pair, contradicting the fact that $\mathbb{B}$ is a nontrivial bisector arrangement. Thus at most two of the lines in $\mathbb{B}$ have an infinite midpoint.

Claim 2. Statement (∗) is valid if at least three bisectors in $\mathbb{B}$ have a finite midpoint.

Without loss of generality, suppose $\mathrm{mid}_{\mathbb{B}}(A), \mathrm{mid}_{\mathbb{B}}(A'), \mathrm{mid}_{\mathbb{B}}(B)$ are all finite. Since $\mathbb{B} \cup \{\mathsf{P}\}$ is a bisector arrangement extending $\mathbb{B}$, this implies C and C' cross each of A, A', B. We claim that at least two of the points $A \cdot C', A' \cdot C', B \cdot C'$ are not equal. Suppose otherwise. Then $\mathrm{mid}_{\mathbb{B}}(B) = \mathrm{mid}_{\mathbb{B}}(C') = A \cdot A'$ since B and C' are bisectors in $\mathbb{B}$ and go through $A \cdot A'$. The line C' bisects the pair B, B' with midpoint $\mathrm{mid}_{\mathbb{B}}(C')$, so since B goes through this point, so does B'. But then all four lines in $\mathbb{B}$ go through the point $\mathrm{mid}_{\mathbb{B}}(C')$, contradicting the fact that $\mathbb{B}$ is a nontrivial bisector arrangement. Thus at least two of the points $A \cdot C', A' \cdot C', B \cdot C'$ are not equal.

We show next that only the line C' can be paired with C if $\mathbb{B} \cup \{\mathsf{P})$ is to be a bisector arrangement. Let $L, M \in \{A, A', B\}$ such that $L \cdot C' \neq M \cdot C'$. Since L bisects $\mathbb{B} \cup \{\mathsf{P}\}$ and C and C' cross L, the point $L \cdot C'$ where C' crosses L is uniquely determined by $\mathrm{mid}_{\mathbb{B}}(L)$ and the point $L \cdot C$ where C crosses L. Similarly, the point $M \cdot C'$ where C' crosses M is uniquely determined by $\mathrm{mid}_{\mathbb{B}}(M)$ and the point $M \cdot C$ where C crosses M. Since $L \cdot C' \neq M \cdot C'$, it follows that, given C, the line C' is the unique line for which $\mathbb{B} \cup \{\mathsf{P}\}$ is a bisector arrangement.

Claim 3. Statement $(*)$ is valid if at least one of the bisectors in $\mathbb{B}$ has an infinite midpoint and at least one has a finite midpoint.

Assume $\mathrm{mid}_{\mathbb{B}}(A) = \infty$. Since A bisects $\mathbb{B} \cup \{\mathsf{P}\}$, A is parallel to at least one of C and C', say C.

We will show C is uniquely determined by C' and $\mathbb{B}$. Since $\mathrm{mid}_{\mathbb{B}}(A)$ is infinite, A is parallel to exactly one of B and B', say B, and so A, B and C are all parallel. By assumption, at least one of the bisectors A, A', B, B' has a finite midpoint in $\mathbb{B}$. Since A, B and C are all parallel, the midpoints of these bisectors cannot be finite, and so at least one bisector $L \in \{A', B'\}$ has a finite midpoint. Let M be the other line in this set. The line L is not parallel to C or C' since $\mathrm{mid}_{\mathbb{B}}(L)$ is finite and L bisects C, C' with this midpoint. Thus, since L crosses both C and C', the point $C \cdot L$ is uniquely determined by $\mathrm{mid}_{\mathbb{B}}(L)$ and $C' \cdot L$. Since C is parallel to A, it follows that C is the unique line for which, given C', $\mathbb{B} \cup \{\mathsf{P}\}$ is a bisector arrangement.

Claim 4. Statement $(*)$ is valid if at most one of the bisectors in $\mathbb{B}$ has an undetermined midpoint.

By Claim 2, if three bisectors in $\mathbb{B}$ have finite midpoints, then statement $(*)$ is valid. If fewer than three have finite midpoints, then by Claim 1 and the assumption that there is at most one undetermined midpoint in $\mathbb{B}$, at least one of the lines in $\mathbb{B}$ has a finite midpoint and at least one has an infinite midpoint, in which case statement $(*)$ is valid by Claim 3.

Claim 5. Statement $(*)$ is valid if exactly two bisectors in $\mathbb{B}$ have undetermined midpoints.

First we claim that at most one of A, A' and at most one of B, B' have an undetermined midpoint. For if $\mathrm{mid}_{\mathbb{B}}(A)$ and $\mathrm{mid}_{\mathbb{B}}(A')$ are undetermined, then (a) $A \in \{B, B'\}$ or A, B and B' are parallel, and (b) $A' \in \{B, B'\}$ or A', B and B' are parallel. If A, B, B' are all parallel, then (b) implies A' is also parallel to these lines, contrary to the assumption that $\mathbb{B}$ is a nontrivial bisector arrangement. Thus $A \in \{B, B'\}$, and similarly, $A' \in \{B, B'\}$, which again contradicts the fact that $\mathbb{B}$ is nontrivial. Thus $\mathrm{mid}_{\mathbb{B}}(A)$ and $\mathrm{mid}_{\mathbb{B}}(A')$ cannot both be undetermined. Similarly, $\mathrm{mid}_{\mathbb{B}}(B)$ and $\mathrm{mid}_{\mathbb{B}}(B')$ cannot both be undetermined.

Without loss of generality, $\mathrm{mid}_{\mathbb{B}}(A)$ and $\mathrm{mid}_{\mathbb{B}}(B)$ are undetermined. By Claim 3, we may reduce to the case that A' and B' both have infinite midpoints or both have finite midpoints. We will rule out the former case. Suppose by way of contradiction that $\mathrm{mid}_{\mathbb{B}}(A')$ and $\mathrm{mid}_{\mathbb{B}}(B')$ are infinite. Then A' is parallel to exactly one of B, B', and B' is parallel to exactly one of A, A'. This implies A' is parallel to B'; A' is not parallel to B; and B' is not parallel to A. However, $\mathrm{mid}_{\mathbb{B}}(A)$ is undetermined, so A must be parallel to B and B' or $A \in \{B, B'\}$. This forces $A = B$, which since A' is parallel to B' contradicts the assumption that $\mathbb{B}$ is a nontrivial bisector arrangement. This shows that $\mathrm{mid}_{\mathbb{B}}(A')$ and $\mathrm{mid}_{\mathbb{B}}(B')$ cannot both be infinite. Therefore, $\mathrm{mid}_{\mathbb{B}}(A')$ and $\mathrm{mid}_{\mathbb{B}}(B')$ are finite.

Next, since $\mathrm{mid}_{\mathbb{B}}(A)$ is undetermined, either A is parallel to B and B' or $A \in \{B, B'\}$. Since $\mathrm{mid}_{\mathbb{B}}(B')$ is finite, A is not parallel to B', and so $A = B$. We claim at least one of the two lines C, C' is not the line $A = B$. Suppose otherwise. The fact that A' and B' bisect C, C' imply $\mathrm{mid}_{\mathbb{B}}(A')$ and $\mathrm{mid}_{\mathbb{B}}(B')$ lie on the line $A = B = C = C'$, which since A' bisects B, B' with midpoint $\mathrm{mid}_{\mathbb{B}}(A')$ implies $\mathrm{mid}_{\mathbb{B}}(A') = \mathrm{mid}_{\mathbb{B}}(B')$, with this point lying on $A = B = C = C'$. But then all four lines A, A', B, B' go through the same point, contrary to the assumption that $\mathbb{B}$ is a nontrivial bisector arrangement. Thus we may assume without loss of generality that C is not the line $A = B$.

We will show C' is uniquely determined by $\mathbb{B}$ and C. Let $\Lambda_{A'}$ denote reflection on the line A' about $\mathrm{mid}_{\mathbb{B}}(A')$, and let $\Lambda_{B'}$ denote reflection on B' about $\mathrm{mid}_{\mathbb{B}}(B')$. Since $\mathbb{B}$ is a nontrivial bisector configuration and $A = B$, the lines A' and B' cannot be parallel. Also, $\Lambda_{A'}$ and $\Lambda_{B'}$ are uniquely determined by $\mathbb{B}$. Since $\Lambda_{A'}(A' \cdot C) = A' \cdot C'$ and $\Lambda_{B'}(B' \cdot C) = B' \cdot C'$, to prove that C' is determined by $\mathbb{B}$ and C, it suffices to show that $A' \cdot C' \neq B' \cdot C'$ since these two points (which are entirely determined by $\mathbb{B}$ and C via the involutions $\Lambda_{A'}$ and $\Lambda_{B'}$) then determine C'. If $A' \cdot C' = B' \cdot C'$, then $A' \cdot C' = B' \cdot C' = A' \cdot B'$, so that

$$\begin{aligned} C \cdot A' = \Lambda_{A'}(A' \cdot C') \ &= \ \Lambda_{A'}(A' \cdot B') \ = \ A' \cdot B \ \in \ B \\ C \cdot B' = \Lambda_{B'}(B' \cdot C') \ &= \ \Lambda_{B'}(A' \cdot B') \ = \ B' \cdot A \ \in \ A. \end{aligned}$$

But since C is not the line $A = B$, this implies A' and B' meet at the same point on $A = B$, contradicting the fact that $\mathbb{B}$ is a nontrivial bisector configuration. Therefore, $A' \cdot C' \neq B' \cdot C'$ and the proof of Claim 5 is complete.

Claim 6. Statement $(*)$ is true in all cases.

If at most one of the bisectors in $\mathbb{B}$ has an undetermined midpoint, then statement $(*)$ is true by Claim 4. If exactly two of these lines has an undetermined midpoint, then statement $(*)$ is true by Claim 5. Thus it suffices to observe that three of the four lines in $\mathbb{B}$ cannot have undetermined midpoints. This is so because in such a case both lines in one of the two pairs A, A' and B, B' must have undetermined midpoints, and, as argued at the beginning of the proof of Claim 5, this leads to a contradiction to the assumption that $\mathbb{B}$ is a nontrivial bisector arrangement. Thus statement $(*)$ is true in all cases, which proves the lemma. □

With Lemma 6.2, we can now prove the main theorem of the article.

Theorem 6.3. *A set of line pairs is a bisector field if and only if it is a nontrivial asymptotic pencil.*

Proof. Suppose that $\mathbb{B}$ is a nontrivial asymptotic pencil. By Proposition 4.5, there is a quadrilateral Q in $\mathbb{B}$ such that $\mathbb{B}$ is the asymptotic pencil of Q. If a line ℓ is in a pair in $\mathbb{B}$, then ℓ bisects Q by Lemma 5.2, and so ℓ bisects $\mathbb{B}$ by Corollary 5.4. Therefore, $\mathbb{B}$ is a bisector arrangement. To see next that $\mathbb{B}$ is a bisector field, let $\mathsf{P} = \{\ell_1, \ell_2\}$ be a pair of lines such that $\mathbb{B} \cup \{\mathsf{P}\}$ is a bisector arrangement. We show $\mathsf{P} \in \mathbb{B}$.

There are two independent reducible quadratics f_1 and f_2 whose zero sets are the pairs of opposite sides of Q. By Lemma 3.3, f_1 and f_2 generate the asymptotic pencil $\mathbb{B}$. Since f_1 and f_2 are reducible conics, every line meets the pairs of lines defined by f_1 and f_2 (possibly at infinity), and so since by assumption ℓ_1 and ℓ_2 bisect $\mathbb{B}$ and hence bisect $\{f_1, f_2\}$, Lemma 5.2 implies there are line pairs P_1 and P_2 in $\mathbb{B}$ such that $\ell_1 \in \mathsf{P}_1$ and $\ell_2 \in \mathsf{P}_2$. Since $\mathbb{B} \cup \{\mathsf{P}\}$ is a bisector arrangement extending $\mathbb{B}$, the collection $\{\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}\}$ is a bisector arrangement extending the bisector arrangement $\{\mathsf{P}_1, \mathsf{P}_2\}$. Since P_1 and P share ℓ_1 and P_2 and P share ℓ_2, Lemma 6.2 implies $\mathsf{P} = \mathsf{P}_1$ or $\mathsf{P} = \mathsf{P}_2$. Thus $\mathsf{P} \in \mathbb{B}$, as claimed.

Conversely, suppose $\mathbb{B}$ is a bisector field. We claim first that $\mathbb{B}$ contains two pairs of lines that are defined by independent quadratics. If $\mathbb{B}$ contains a degenerate parabola defined by a quadratic f_1, then since $\mathbb{B}$ is a nontrivial bisector arrangement, there is a reducible quadratic f_2 with zero set in $\mathbb{B}$ such that not all four linear components of f_1 and f_2 are parallel. Thus f_1 and f_2 are independent. If $\mathbb{B}$ does not contain a degenerate parabola, then choose f_1 to be any degenerate hyperbola with zero set in $\mathbb{B}$ and, using the fact that $\mathbb{B}$ is a nontrivial bisector arrangement, choose f_2 to be a degenerate hyperbola in $\mathbb{B}$ such that $V(f_1) \neq V(f_2)$. If f_1 is not a translation of f_2, then f_1 and f_2 are independent quadratics defining line pairs in $\mathbb{B}$. If, however, f_2 is a translation of f_1, then since $\mathbb{B}$ is nontrivial and by assumption consists only of hyperbolas, there is a hyperbola f_3 that is not a translation of f_1 or f_2. Choose whichever of f_1 and f_2 does not share a center with f_3, and this quadratic and f_3 will be independent quadratics that define line pairs in B.

In all cases, we have found a pair of independent quadratics f_1 and f_2 that define line pairs in $\mathbb{B}$ and have the additional property that if f_1 and f_2 are hyperbolas, then f_1 and f_2 do not share the same center. This implies the line pairs defined by f_1 and f_2 form a nontrivial bisector arrangement contained in $\mathbb{B}$.

We claim $\mathbb{B}$ is the asymptotic pencil generated by f_1 and f_2. Let f be a quadratic that defines a line pair in $\mathbb{B}$. The line pairs that are the zero sets of f_1, f_2 and f form a bisector arrangement, and so by Lemma 5.2, the linear components of f are linear components of reducible quadratics g_1 and g_2 in $\Bbbk f_1 + \Bbbk f_2 + \Bbbk$. We have established in the first part of the proof that the set of reduced conics in $\Bbbk f_1 + \Bbbk f_2 + \Bbbk$, i.e., the asymptotic pencil generated by f_1 and f_2, is the set of line pairs of a bisector field. Thus the line pairs defined by f_1, f_2, g_1 form a bisector arrangement, as do those of f_1, f_2, g_2. Since the bisector arrangement consisting of the line pairs of f_1 and f_2 is nontrivial and f and g_1 share a linear component, and f and g_2 share the other linear component of f, Lemma 6.2 implies $V(f) = V(g_1)$ or $V(f) = V(g_2)$. Either way, $f \in \Bbbk f_1 + \Bbbk f_2 + \Bbbk$, and hence the pair of lines defined by f is in the asymptotic pencil generated by f_1, f_2, which shows the pairs of lines in $\mathbb{B}$ are in this asymptotic pencil. Therefore, since $\mathbb{B}$ is a bisector field contained in a bisector arrangement consisting of the set of line pairs in the asymptotic pencil generated by f_1, f_2, the maximality of $\mathbb{B}$ implies these two arrangements are equal, which proves the theorem. □

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References

[1] Beltrami, E.: Intorno ad alcuni teoremi di Feuerbach e di Steiner. Esercitazione analitica. Memorie R. Accad. Sci. Ist. Cl. Sci. fis. Bologna, serie III, tomo V, (1874), 543–566
[2] Berele, A., Catoiu, S.: Bisecting envelopes of convex polygons. Adv. Appl. Math. **137**, 102342 (2022)
[3] Berele, A., Catoiu, S.: Extreme area and perimeter bisectors in a triangle. Math. Aequat. (2023). https://doi.org/10.1007/s00010-023-01001-9
[4] Berger, M.: Geometry II, Springer-Verlag (1987)
[5] Olberding, B., Walker, E.: Bisector fields of quadrilaterals, Beitr. Algebra Geom., https://doi.org/10.1007/s13366-023-00728-5
[6] Olberding, B., Walker, E.: Bisector fields and projective duality, submitted, https://doi.org/10.48550/arXiv.2306.08612
[7] Vaccaro, M.: Historical origins of the nine-point conic-the contribution of Eugenio Beltrami. Hist. Math. **51**, 26–48 (2020)

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